

Matter Correlations in Branched Polymers

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Abstract

We analyze correlation functions in a toy model of a random geometry interacting with matter. We show that in general the connected correlator will contain a long-range scaling part. This result supports the previously conjectured general form of correlation functions on random geometries. We discuss the interplay between matter and geometry and the role of the symmetry in the matter sector.

Introduction

In theories with fluctuating (quantized) geometry it is non-trivial to define the concept of a connected correlation function. This is especially apparent in the approaches like dynamical triangulation (DT) which are formulated in a non-perturbative, coordinate-free way and permit to go beyond the flat background expansion. In such formulations the distance itself is a dynamical and a highly non-local object. This leads to conceptual and technical difficulties when trying to generalize the standard fixed-geometry definitions [1, 2].

Due to the non-locality the correlation functions are very difficult to study and very few analytic results exist [3, 4]. A growing body of numerical evidence suggests however the existence of a simple structure common to all correlators in various ensembles of random geometries [3]. This structure can be studied with toy models.

In this contribution we analyze a simple model of random geometry interacting with matter which permits the detailed analytical analysis of the connected and disconnected correlators.

The structure of this paper is as follows : we first introduce the model, then we derive its thermodynamical properties and general structure of the correlation functions in grand-canonical and canonical ensembles. As an application of the developed formalism we then investigate the case of Ising spins in magnetic field. We compare our results with the Monte-Carlo simulations.

Branched Polymers

Branched Polymers (BP) provide a very simple but non-trivial example of a random geometry ensemble [5]. Moreover they exhibit a wide range of properties in common with higher dimensional DT systems [6].

By Branched Polymers we understand the ensemble of *planar, labeled* trees (\mathcal{T}) [5, 7]. Each tree T is weighted with a factor $\rho(T, X)$ depending on the geometry and possibly on the additional “matter” fields X living on the vertices of the tree. Partition function of the model is thus defined to be :

$$Z(\mu) = \sum_{T \in \mathcal{T}} \sum_X e^{-\mu n} \rho(T, X) \quad (1)$$

where $n = n(T)$ denotes the number of vertices in the given tree. Considering the most general form of two-point actions for geometry and matter fields we take ρ to be :

$$\rho(T, X) = \prod_i p_{q_i, x_i} \prod_{\langle i, j \rangle} g_{q_i, x_i; q_j, x_j}. \quad (2)$$

Here q_i and x_i denote respectively the number of branches emerging from the vertex i and the value of the matter field(s) in this vertex, $\langle i, j \rangle$ denotes the set of all nearest neighbors pairs. Non-negative coefficients $p_{q, x}$ and $g_{q, x; q', x'}$ are parameters of the model. The only restriction is that we assume $g_{q, x; q', x'}$ to be symmetric *ie* $g_{q, x; q', x'} = g_{q', x'; q, x}$. The canonical partition function Z_n is related to (1) through the discrete Laplace transform :

$$Z(\mu) = \sum_{n=1}^{\infty} e^{-\mu n} Z_n. \quad (3)$$

All the properties of this ensemble can be obtained from the partition function of the ensemble \mathcal{T}_{pl} of *rooted, planted* trees *ie* the trees with one point (root) marked with the condition that this marked vertex has only one branch [7, 8]. This vertex thus constitutes a “handle” by which planted trees can be glued together. It is convenient to assign to this vertex a dummy number of branches q_0 and matter field x_0 . In this way we obtain a whole set of partition functions for the planted ensemble :

$$Z_{q_0, x_0}(\mu) = \sum_{T \in \mathcal{T}_{pl}} \sum_X e^{-\mu n} \prod_{i=1}^n p_{q_i, x_i} \prod_{\langle i, j \rangle} g_{q_i, x_i; q_j, x_j}. \quad (4)$$

Here n denotes the number of vertices not including the root. The sum over all neighbors includes the root (denoted by index zero).

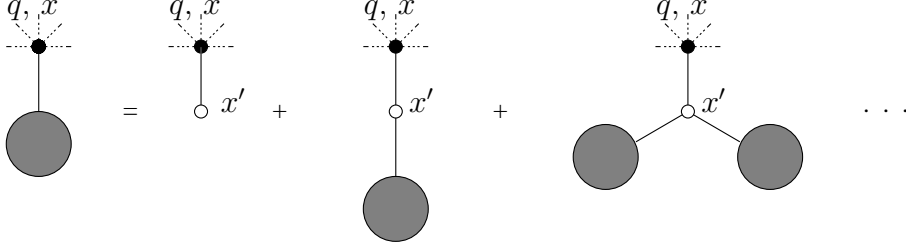


Figure 1: Recursive construction of the partition function $Z_{q,x}(\mu)$ of the ensemble of rooted planted trees.

Critical behavior of the model is characterized by singularities of the functions $Z_{q,x}(\mu)$. To find them we first observe that those functions fulfill a set of equations (see figure 1)[5] :

$$Z_{q,x}(\mu) = e^{-\mu} \sum_{q',x'} g_{q,x;q',x'} p_{q',x'} Z_{q',x'}^{q'-1}(\mu). \quad (5)$$

Expanding $Z_{q,x}(\mu)$ around μ :

$$Z_{q,x}(\mu + \Delta\mu) \approx Z_{q,x}(\mu) + \partial_\mu Z_{q,x}(\mu) \Delta\mu \quad (6)$$

and inserting into (5) we obtain

$$\sum_{q',x'} (\delta_{q,x;q',x'} - M_{q,x;q',x'}(\mu)) Z_{q',x'}'(\mu) = -Z_{q,x}(\mu) \quad (7)$$

where operator \mathbf{M} is defined by :

$$M_{q,x;q',x'}(\mu) = e^{-\mu} g_{q,x;q',x'} p_{q',x'} (q' - 1) Z_{q',x'}^{q'-2}(\mu). \quad (8)$$

Linear system (7) has an unique solution provided that the operator $\mathbf{1} - \mathbf{M}(\mu)$ is invertible. The condition

$$\mathbf{1} - \mathbf{M}(\mu_0) \text{ is not invertible} \quad (9)$$

defines the critical point μ_0 and is equivalent to the statement that operator $\mathbf{M}(\mu_0)$ has an eigenvalue $\lambda^{(0)} = 1$. As we will see later, this corresponds to an *elongated* or *tree* phase[9].

Around point μ_0 expansion (6) is not valid. Going beyond the linear approximation, the first non-vanishing term will be in general a quadratic

form¹ leading to :

$$Z_{q,x}(\mu_0 + \Delta\mu) \approx Z_{q,x}(\mu_0) - \Delta Z_{q,x}(\mu_0) \sqrt{\Delta\mu}. \quad (10)$$

As we show in the appendix A $\Delta Z_{q,x}(\mu_0)$ is the eigenvector associated with the eigenvalue $\lambda^{(0)} = 1$. We have chosen the sign in (10) as to make the $\Delta Z_{q,x}(\mu_0)$ positive. This is ensured by the fact that $Z_{q,x}(\mu)$ decreases as μ is increased as seen from (4).

Condition (9) is not the only way the singularity of the partition function can arise. The function of $Z_{q,x}(\mu)$ defined by possibly infinite series on the right-hand side of (5) can develop singularities. These singularities will depend on the asymptotic behavior of p 's. This scenario corresponds to *crumpled* or *bush* phase. When both conditions are met simultaneously *ie* $\lambda^{(0)} = 1$ and right-hand side of (5) is singular we may have a *marginal* phase [8, 9].

To calculate expectation values of a given operator \mathbf{A} we introduce another kind of a partition function :

$$G^{\mathbf{A}}(\mu) = \sum_{T \in \mathcal{T}} \sum_X e^{-\mu n} \rho(T, X) \sum_{i \in T} A_{q_i, x_i} \quad (11)$$

which can be viewed as the partition function of the ensemble of trees with one point (root) marked, with operator \mathbf{A} inserted at the root :

$$G^{\mathbf{A}}(\mu) = \sum_{T \in \mathcal{T}_{root}} \sum_X e^{-\mu n} \rho(T, X) A_{q_0, x_0}. \quad (12)$$

We can express $G^{\mathbf{A}}(\mu)$ by the functions $Z_{q,x}(\mu)$ (see figure 2 and reference [7] for the relation between planted and rooted ensembles) :

$$G^{\mathbf{A}}(\mu) = \sum_x \sum_{q=1}^{\infty} A_{q,x} p_{q,x} \frac{1}{q} Z_{q,x}^q(\mu). \quad (13)$$

Assuming that we are in the elongated phase we can expand the above expression using (10) to obtain :

$$\begin{aligned} G^{\mathbf{A}}(\mu) &\approx \sum_x \sum_{q=1}^{\infty} A_{q,x} p_{q,x} Z_{q,x}^q(\mu_0) \frac{1}{q} \left(1 - q \frac{\Delta Z_{q,x}(\mu_0)}{Z_{q,x}(\mu_0)} \sqrt{\Delta\mu} \right) \\ &\approx \sum_x \sum_{q=1}^{\infty} A_{q,x} p_{q,x} Z_{q,x}^q(\mu_0) \frac{1}{q} \exp \left(-q \frac{\Delta Z_{q,x}(\mu_0)}{Z_{q,x}(\mu_0)} \sqrt{\Delta\mu} \right). \end{aligned} \quad (14)$$

¹ Allowing negative values for p 's it is possible to make also higher term of the expansion vanish and by this tune the system to the multicritical point where the corrections depend on $\Delta\mu^{\frac{1}{k}}$ [10].

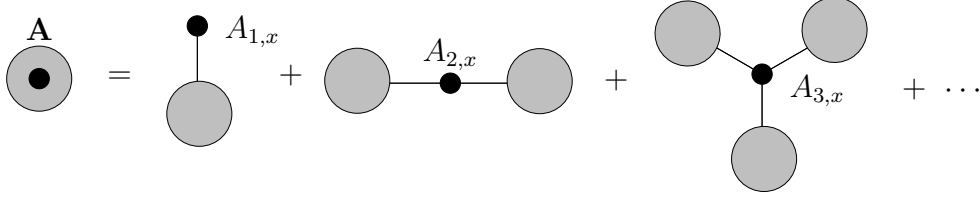


Figure 2: Rooted ensemble.

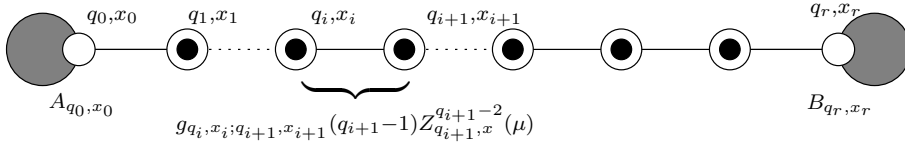


Figure 3: Correlation function

Performing the inverse Laplace transform we obtain the large n behavior of the canonical function :

$$G_n^{\mathbf{A}} \approx \frac{C}{n^{\frac{3}{2}}} e^{\mu_0 n} \sum_x \sum_{q=1}^{\infty} A_{q,x} p_{q,x} Z_{q,x}^{q-1}(\mu_0) \Delta Z_{q,x}(\mu_0). \quad (15)$$

We get the expectation value of operator $\langle \mathbf{A} \rangle_n = \frac{1}{n} \langle \sum_i A_{q_i, x_i} \rangle_n$ in canonical ensemble by normalizing the function $G_n^{\mathbf{A}}$:

$$\langle \mathbf{A} \rangle_n = \frac{G_n^{\mathbf{A}}}{G_n^{\mathbf{1}}} \stackrel{N \rightarrow \infty}{\approx} \frac{\sum_x \sum_{q=1}^{\infty} \mathbf{A}_{q,x} p_{q,x} Z_{q,x}^{q-1}(\mu_0) \Delta Z_{q,x}(\mu_0)}{\sum_x \sum_{q=1}^{\infty} p_{q,x} Z_{q,x}^{q-1}(\mu_0) \Delta Z_{q,x}(\mu_0)}. \quad (16)$$

Correlation functions

We define the integrated (unnormalized) correlation functions of operators \mathbf{A} and \mathbf{B} as :

$$G_{\mu}^{\mathbf{AB}}(r) = \sum_{T \in \mathcal{T}} \sum_X \rho(T, X) \sum_{i,j \in T} A_{q_i, x_i} B_{q_j, x_j} \delta_{d(i,j), r}. \quad (17)$$

The distance $d(i, j)$ is the *geodesic* distance ie the length of the shortest path between points i and j . In our case this definition is especially simple as only one path exists between two points on a tree.

To calculate the functions (17) we consider the chain of $r+1$ points along the path joining points i and j and we sum over all possible values of the

coordination numbers and matter fields values in each vertex (see figure 3). This can be done by the transfer matrix technique. In addition to one-point weight each vertex inside the chain with coordination number q has $q - 2$ trees attached to it. Those trees can be arranged in $q - 1$ ways relatively to the chain. This gives an additional $(q - 1)Z_{q,x}^{q-2}(\mu)$ factor per vertex [10, 11]. If in constructing the transition matrix we assign the one-point weights to the right-hand side, we end up with a matrix that is *identical* to the operator \mathbf{M} defined by (8) except that the first row and column are zero as vertices with order one cannot appear inside the chain (see figure 3). We will call this truncated matrix $\widetilde{\mathbf{M}}$.

In terms of matrix $\widetilde{\mathbf{M}}$ the correlation function (17) can be expressed as :

$$G_{\mu}^{\mathbf{AB}}(r) = \sum_{q,x} \sum_{q',x'} Z_{q,x}(\mathbf{A}) (\mathbf{M} \widetilde{\mathbf{M}}^{r-2} \mathbf{M})_{q,x;q',x'} \frac{Z_{q',x'}(\mathbf{B})}{(q' - 1)Z_{q',x}^{q'-2}(\mu)} \quad (18)$$

with

$$Z_{q,x}(\mathbf{A}) = e^{-\frac{1}{2}\mu_0} A_{q,x} p_{q,x} Z_{q,x}^{q-1}(\mu). \quad (19)$$

The functions $Z_{q,x}(\mathbf{A})$ account for the end-point effects and the apparent asymmetry is compensated by the asymmetry of matrix the $\widetilde{\mathbf{M}}$. The above expression can be expanded in the basis of the eigenvectors of the matrix \mathbf{M} (see appendix B for details) :

$$G_{\mu}^{\mathbf{AB}}(r) = \sum_k (\lambda^{(k)})^r \sum_{q,x} Z_{q,x}(\mathbf{A}) v_{q,x}^{(k)} \sum_{q',x'} Z_{q',x'}(\mathbf{B}) v_{q',x'}^{(k)}. \quad (20)$$

The vectors $\mathbf{v}^{(k)}$ are the right eigenvectors of the matrix $\mathbf{M}(\mu)$. They normalization is defined through the relations (57) and (60) in appendix B.

The formula (20) is exact and valid for any BP system with weights defined by (2). We will now expand this expression around μ_0 assuming that we are in elongated phase. From (10) we expect :

$$\begin{aligned} \lambda_k &\approx \lambda_0^{(k)} - \lambda_1^{(k)} \sqrt{\Delta\mu}, \\ \sum_{q,x} Z_{q,x}(\mathbf{A}) v_{q,x}^{(k)} &\approx A_0^{(k)} - A_1^{(k)} \sqrt{\Delta\mu}. \end{aligned} \quad (21)$$

As we have shown in the previous section, in elongated phase, $\lambda_0^{(0)} = 1$ and $v_{q,x}^{(0)} \propto \Delta Z_{q,x}(\mu_0)$. Comparing (21) with (15) and (16) we find that $A_0^{(0)} = \mathbf{1}_0^{(0)} \langle \mathbf{A} \rangle$ (by $\mathbf{1}$ we denote a constant operator equal to unity : $1_{q,x} = 1$).

Inserting (21) into (20) and using approximation $(1-x)^r \approx e^{-x \cdot r}$ we finally obtain the structure of an arbitrary correlation function in the elongated phase in the limit $\Delta\mu \rightarrow 0$ with $\sqrt{\Delta\mu} \cdot r = \text{const} \lesssim 1$:

$$G_\mu^{\mathbf{AB}}(r) \approx (\mathbf{1}_0^{(0)})^2 \langle \mathbf{A} \rangle \langle \mathbf{B} \rangle e^{-a^{(0)}(r+\delta_{\mathbf{A}}^{(0)}+\delta_{\mathbf{B}}^{(0)})\sqrt{\Delta\mu}} + \sum_{k>0} A_0^{(k)} B_0^{(k)} (\lambda_0^{(k)})^r e^{-a^{(k)}(r+\delta_{\mathbf{A}}^{(k)}+\delta_{\mathbf{B}}^{(k)})\sqrt{\Delta\mu}} \quad (22)$$

where

$$a^{(k)} = \frac{\lambda_1^{(k)}}{\lambda_0^{(k)}}, \quad \text{and} \quad \delta_{\mathbf{A}}^{(k)} = \frac{1}{a^{(k)}} \frac{A_1^{(k)}}{A_0^{(k)}} \quad (23)$$

At this stage we want to make two remarks : i) The first term in (22) is a function of only one variable $x = (r+\delta_{\mathbf{A}}^{(0)}+\delta_{\mathbf{B}}^{(0)})\sqrt{\Delta\mu}$ and so the distance scale is set by $\frac{1}{\sqrt{\Delta\mu}}$. As we approach the critical value μ_0 (thermodynamic limit) the size of the system becomes infinite. This justifies the name elongated phase. In this phase Hausdorff dimension $d_H = 2$ (this will be more apparent in the canonical formulation). In thermodynamic limit shifts $\delta_{\mathbf{A}(\mathbf{B})}^{(0)}$ can be neglected but they are necessary to maintain the scaling at any finite volume. This fact was first observed in MC simulations of 2D random surfaces [12].

ii) If there exists a finite gap between $\lambda_0^{(0)} = 1$ and the “next” eigenvalue $\lambda = \limsup_{k>0} \lambda_0^{(k)}$ then the first term will dominate at large distances :

$$G^{\mathbf{AB}}(r) \approx \langle \mathbf{A} \rangle \langle \mathbf{B} \rangle G^{\mathbf{11}}(r + \delta_{\mathbf{A}}^{\text{eff.}} + \delta_{\mathbf{B}}^{\text{eff.}}) \quad \text{for} \quad r \gg \xi \equiv -\frac{1}{\log \lambda} \quad (24)$$

with

$$\delta_{\mathbf{A}}^{\text{eff.}} = \delta_{\mathbf{A}}^{(0)} - \delta_1^{(0)}. \quad (25)$$

The function $G^{\mathbf{11}}(r) \equiv G(r)$ contains no operator insertions. Properly normalized it gives average volume of the spherical shell of radius r . As such it encodes the general information about the average shape and size of the system. The formula (24) states that the whole effect of inserting operators is limited to multiplicative factors and shifts of the argument. The shifts are additive which means they depend on each operator separately.

Obviously the formulas (22) and (23) are valid only when $A_0^{(k)}$, $\lambda_0^{(k)}$ and $\lambda_1^{(k)}$ are non-zero. We will assume that this is the case for λ 's and discuss the case of $A_0^{(k)} = 0$. Expanding (21) to the next order :

$$\sum_{q,x} Z_{q,x}(A) v_{q,x}^{(k)} \approx -A_1^{(k)} \sqrt{\Delta\mu} + A_2^{(k)} \Delta\mu \quad (26)$$

we find the corresponding term in the sum (22) to be :

$$-A_1^{(k)}\sqrt{\Delta\mu}\exp\left(-a^{(k)}\left(r+\frac{A_2^{(k)}}{A_1^{(k)}}\frac{1}{a^{(k)}}\right)\sqrt{\Delta\mu}\right)=$$

$$A_1^{(k)}\frac{1}{a^{(k)}}\frac{\partial}{\partial r}\exp\left(-a^{(k)}\left(r+\frac{A_2^{(k)}}{A_1^{(k)}}\frac{1}{a^{(k)}}\right)\sqrt{\Delta\mu}\right). \quad (27)$$

We will use this result in the next section when discussing the connected correlators.

Connected correlators

To study the connected correlators we switch now to the canonical ensemble. Connected correlation functions $G_n^{\mathbf{AB}}(r)$ are defined in the same way as grand-canonical ones, but now summation in (17) is restricted to the canonical ensemble. Additionally we now normalize correlation functions so that $\sum_r G_n^{\mathbf{11}}(r) \propto n$. Taking the inverse Laplace transform off (22) we obtain :

$$G_n^{\mathbf{AB}}(r) \approx 2\sqrt{n}(\mathbf{1}_0^{(0)})^2 \langle \mathbf{A} \rangle \langle \mathbf{B} \rangle g(a^{(0)} \frac{r + \delta_{\mathbf{A}}^{(0)} + \delta_{\mathbf{B}}^{(0)}}{2\sqrt{n}})$$

$$+ 2\sqrt{n} \sum_{k=1} A_0^{(k)} B_0^{(k)} (\lambda_0^{(k)})^r g(a^{(k)} \frac{r + \delta_{\mathbf{A}}^{(k)} + \delta_{\mathbf{B}}^{(k)}}{2\sqrt{n}}) \quad (28)$$

where $g(x) = xe^{-x^2}$. The structure of the correlation functions is thus the same as for the grand-canonical ensemble with n playing the role of $1/\Delta\mu$ and $g(x)$ substituted for e^{-x} . In particular the asymptotic form (24) is also valid in this ensemble.

We define the connected correlator by² :

$$G^{\mathbf{AB}conn} = G_n^{\mathbf{A}^{conn}\mathbf{B}^{conn}}(r) = G_n^{(\mathbf{A}-\langle\mathbf{A}\rangle)(\mathbf{B}-\langle\mathbf{B}\rangle)}(r)$$

$$= G_n^{\mathbf{AB}}(r) - \langle \mathbf{A} \rangle G_n^{\mathbf{1B}}(r) - \langle \mathbf{B} \rangle G_n^{\mathbf{1A}}(r) + \langle \mathbf{A} \rangle \langle \mathbf{B} \rangle G_n^{\mathbf{11}}(r). \quad (29)$$

The operator $A_{q,x}^{con} = A_{q,x} - \langle \mathbf{A} \rangle$ has a zero average. It is easy to check that :

$$A_1^{(1)conn} = a^{(0)}\mathbf{1}_0^{(0)}\langle \mathbf{A} \rangle \delta_{\mathbf{A}}^{eff} \quad \text{and} \quad A_0^{(k)conn} = A_0^{(k)} - \langle \mathbf{A} \rangle \mathbf{1}_0^{(k)}$$

$$A_1^{(k)conn} = A_1^{(k)} - \langle \mathbf{A} \rangle \mathbf{1}_1^{(k)} \quad (30)$$

²See [1] and [2] for discussion of other possible definitions.

and thence from formula (27) we obtain :

$$G_n^{\mathbf{AB}conn}(r) \approx \frac{\delta_{\mathbf{A}}^{eff.} \delta_{\mathbf{B}}^{eff.}}{2\sqrt{n}} \mathbf{1}_{0,0}^2 \langle \mathbf{A} \rangle \langle \mathbf{B} \rangle a_0^2 g''(a^{(0)} \frac{r + \dots}{2\sqrt{n}}) \\ + 2\sqrt{n} \sum_{k=1} \mathbf{1}_{k,0}^2 \left(\frac{A_0^{(k)}}{\mathbf{1}_0^{(k)}} - \langle \mathbf{A} \rangle \right) \left(\frac{B_0^{(k)}}{\mathbf{1}_0^{(k)}} - \langle \mathbf{B} \rangle \right) (\lambda^{(k)})^r g(a^{(k)} \frac{r + \bar{\delta}_{\mathbf{A}}^{(k)} + \bar{\delta}_{\mathbf{B}}^{(k)}}{2\sqrt{n}}). \quad (31)$$

The dots stand for the shifts which depend on $A_2^{(0)}$ and $B_2^{(0)}$ (see (27)). The shifts $\bar{\delta}_{\mathbf{A}(\mathbf{B})}^{(k)}$ are obtained by inserting (30) into (23).

A more intuitive way of obtaining this results is to insert (22) into the expanded expression for the connected correlator (29)[3, 13].

Comparing (31) with (28) we see that the scaling long-range part is suppressed by factor of n . We will call this weak long-range correlations³. The non-scaling terms remain of the leading-order in n and for any finite r will eventually dominate in the thermodynamic limit. However when considered as functions of the scaling variable $x = r/\sqrt{n}$ they will vanish in this limit.

The weak long-range correlations can be further suppressed by using an “improved” version of the connected correlator [3] :

$$G_n^{\mathbf{AB}conn}(r) = G_n^{\mathbf{AB}}(r) - \langle \mathbf{A} \rangle_n G_n^{\mathbf{1B}}(r + \delta_{\mathbf{A}}^{eff.}) - \langle \mathbf{B} \rangle G_n^{\mathbf{1A}}(r + \delta_{\mathbf{B}}^{eff.}) \\ + \langle \mathbf{A} \rangle \langle \mathbf{B} \rangle G_n^{\mathbf{11}}(r + \delta_{\mathbf{A}}^{eff.} + \delta_{\mathbf{B}}^{eff.}). \quad (32)$$

Inserting this into (29) and omitting the shifts we obtain :

$$G_n^{\mathbf{AB}conn}(r) \approx \frac{1}{n} C g'''(a^{(0)} \frac{r}{2\sqrt{n}}) \\ + 2\sqrt{n} \sum_k \mathbf{1}_{k,0}^2 \left(\frac{A_0^{(k)}}{\mathbf{1}_0^{(k)}} - \langle \mathbf{A} \rangle \lambda_k^{\delta_{\mathbf{A}}} \right) \left(\frac{B_0^{(k)}}{\mathbf{1}_0^{(k)}} - \langle \mathbf{B} \rangle \lambda_k^{\delta_{\mathbf{B}}} \right) (\lambda_0^{(k)})^r g(a^{(k)} \frac{r}{2\sqrt{n}}). \quad (33)$$

We see that the scaling term is further suppressed by a factor \sqrt{n} while the non-scaling terms remain of the leading-order.

Ising model

In this section we consider the Ising model on BP as defined in [15]. This corresponds to the choice of $g_{q,s;q's'} = e^{\beta s \cdot s'}$ and $p_{q,s} = p_q \cdot e^{-hs}$. Spins take

³This is in analogy with the model of spins interacting with infinite range forces but with strength diminishing with the system size and leading to similar behavior of the connected correlation functions $G_n^{conn}(r) \sim \frac{f(r)}{n}$ [14].

values ± 1 . Because we have no geometric interactions, partition function $Z_{q,x}(\mu)$ does not depend on q . Summing the right-hand side of (5) over q' we obtain [15] :

$$\begin{aligned} Z_+(\mu) &= e^{-\mu} \left(e^{\beta} e^h F(Z_+(\mu)) + e^{-\beta} e^{-h} F(Z_-(\mu)) \right) \\ Z_-(\mu) &= e^{-\mu} \left(e^{-\beta} e^h F(Z_+(\mu)) + e^{\beta} e^{-h} F(Z_-(\mu)) \right) \end{aligned} \quad (34)$$

where :

$$F(Z) = \sum_{q=1}^{\infty} p(q) Z^{q-1} \quad (35)$$

The critical point condition (9) is now :

$$\begin{aligned} 1 &= e^{-\mu_0} e^{\beta} \left(e^h F'(Z_+(\mu_0)) + e^{-h} F'(Z_-(\mu_0)) \right) \\ &\quad - e^{-2\mu_0} 2 \sinh(2\beta) F'(Z_+(\mu_0)) F'(Z_-(\mu_0)). \end{aligned} \quad (36)$$

The transfer matrix \mathbf{M} (there is no distinction between $\widetilde{\mathbf{M}}$ and \mathbf{M} as we no longer have the dependency on q) is :

$$\mathbf{M} = e^{-\mu} \begin{pmatrix} e^{\beta} e^h F'(Z_+) & e^{-\beta} F'(Z_-) \\ e^{-\beta} F'(Z_+) & e^{\beta} e^{-h} F'(Z_-) \end{pmatrix} \quad (37)$$

From (31) we have :

$$\begin{aligned} G_n^{\mathbf{AB}conn}(r) &\approx \frac{1}{2\sqrt{n}} X_{\mathbf{AB}} g''(a^{(0)} \frac{r + \dots}{2\sqrt{n}}) \\ &\quad + 2\sqrt{n} Y_{\mathbf{AB}} (\lambda_0^{(1)})^r g(a^{(1)} \frac{r + \dots}{2\sqrt{n}}) \end{aligned} \quad (38)$$

We will refer to the first term of the above expression as to “scaling” term and to the other as to “non-scaling” one. The expressions for $X(Y)_{\mathbf{AB}}$ can be readout from (31). The case of the “curvature-curvature” correlator⁴ $G_n^{\mathbf{q}\mathbf{q}}(r)$ is special (operator \mathbf{q} is defined by $q_{u,x} = u$). We can obtain it directly from $G_n^{\mathbf{1}\mathbf{1}}(r)$ using the relations :

$$\begin{aligned} G^{\mathbf{A}\mathbf{q}}(r) &= G^{\mathbf{A}\mathbf{1}}(r+1) + G^{\mathbf{A}\mathbf{1}}(r), \quad r > 0 \\ G^{\mathbf{A}\mathbf{q}}(0) &= G^{\mathbf{A}\mathbf{1}}(1) \quad \text{and} \quad \langle \mathbf{q} \rangle = 2 \end{aligned} \quad (39)$$

⁴In DT the curvature is related to the number of D -simplices incident on a $(D-2)$ -simplex

that are satisfied on any tree. Inserting the above relations into (29) we obtain :

$$G_n^{\mathbf{q}\mathbf{q}conn}(r) = G^{\mathbf{1}\mathbf{1}}(r+2) - 2G^{\mathbf{1}\mathbf{1}}(r+1) + G^{\mathbf{1}\mathbf{1}}(r) \approx G''^{\mathbf{1}\mathbf{1}}(r+1) \quad (40)$$

When $h = 0$ the symmetry implies $Z_+(\mu) = Z_-(\mu)$ and the equations (5) decouple. The only effect of the presence of Ising spins is to renormalize the chemical potential [15]. This leads to :

$$G_n^{\mathbf{ss}}(r) = G_n^{\mathbf{ss}conn}(r) = (\tanh \beta)^r \mathcal{G}_n^{\mathbf{1}\mathbf{1}}(r) \quad (41)$$

where

$$\mathcal{G}_n^{\mathbf{1}\mathbf{1}}(r) \approx C\sqrt{n}re^{-a\frac{r^2}{n}} \quad (42)$$

is the point-point correlation function of the pure (without spins) BP model. The correlation functions of the operators depending only on the geometry remain unaffected by the presence of spins. We see that in this special case matter and geometry sectors do not interact :

$$X_{\mathbf{ss}} = 0 \quad \text{and} \quad Y_{\mathbf{q}\mathbf{q}} = 0. \quad (43)$$

The vanishing of $X_{\mathbf{ss}}$ is actually a general feature : whenever symmetry enforces the condition $Z_{q,x}(\mu) = Z_q(\mu)$ the matter-matter correlators will not contain the long-range scaling part : $X_{xx} = 0$ (see Appendix C for the proof).

When $h \neq 0$ the symmetry is explicitly broken and we have to investigate the full system (34,36). Those equations cannot be solved analytically even for the simplest choices of p 's. They are however not difficult to solve numerically. Once μ_0 and $Z_{\pm}(\mu_0)$ are known one can calculate other necessary quantities. To obtain the $\sqrt{\Delta\mu}$ order terms the standard perturbation theory can be used.

For simplicity we have chosen the weights $p_1 = p_2 = p_3 = 1$ and all others equal to zero. The inverse temperature and magnetic field were set to $\beta = 1.0$ and $h = 0.1$. For those values we obtained $\lambda_0^{(1)} \approx 0.5709$ corresponding to spin-spin correlation length $\zeta \approx 1.785$. For $a^{(0)}$ and $a^{(1)}$ we obtained respectively 1.1795 and 0.5760. We have compared those theoretical large n limit predictions with the results of MC simulations of the systems with 250, 1000 and 4000 vertices.

First we have tested the validity of the approximation (24). For $r \gg \zeta$ we expect

$$G^{\mathbf{1}\mathbf{s}}(r) \approx \langle \mathbf{s} \rangle G(r + \delta_{\mathbf{s}}^{eff.}) \quad \text{and} \quad G^{\mathbf{ss}}(r) \approx \langle \mathbf{s} \rangle G^{\mathbf{s}}(r + \delta_{\mathbf{s}}^{eff.}) \quad (44)$$

n <i>range</i>	$\langle \mathbf{s} \rangle$	$\delta_s^{eff.}$	E^2
<i>theor.</i>	0.6980	-0.6533	
250	0.6961(3)	-0.647(4)	0.20e-3
10-80	0.6962(3)	-0.625(4)	0.90e-4
1000	0.6979(1)	-0.644(1)	0.10e-2
10-160	0.6979(1)	-0.652(1)	0.15e-3
4000	0.69798(7)	-0.649(1)	0.56e-4
10-200	0.69799(7)	-0.652(1)	0.26e-3

Table 1: Fitted values of shifts and average spin values.

In the table 1 we list the measured shifts and mean magnetization obtained by fitting the formula above to the measured values of $G_n^{\mathbf{1s}}(r)$ and $G_n^{\mathbf{ss}}(r)$. The fitting was done by interpolating the functions by cubic splines and then minimizing the error function :

$$E = \sum_r (G_n^{\mathbf{AB}}(r) - \langle \mathbf{A} \rangle G_n^{\mathbf{1B}}(r - \delta_{\mathbf{A}}^{eff.}))^2 \quad (45)$$

with $\delta_{\mathbf{A}}^{eff.}$ and $\langle \mathbf{A} \rangle$ as free parameters. As we can see the formulas (44) are very well satisfied already for the small system sizes.

For the connected spin-spin correlator we obtain :

$$X_{\mathbf{ss}} = 0.1414 \quad \text{and} \quad Y_{\mathbf{ss}} = 0.555. \quad (46)$$

The MC results for this correlator are plotted in the figure 4. As expected at short distances it is dominated by non-scaling correlations. The line denotes the non-scaling part of (38) in the limit of $n \rightarrow \infty$. In the inlay we have emphasized the scaling part by plotting the correlators scaled by $2\sqrt{n}$ as the function of the scaling variable $x = \frac{r}{2\sqrt{n}}$. The line denotes the scaling part of (38).

For connected curvature-curvature correlator we obtain :

$$X_{\mathbf{qq}} = 0.6861 \quad \text{and} \quad Y_{\mathbf{qq}} = 0.0070. \quad (47)$$

We see that in this case the non-scaling part is strongly suppressed. This can be seen on the figure 5 where we have again shown the rescaled functions. The non-scaling part can be seen only by looking at small r and large n . In the inlay we show the small r region. The continuous line denotes the

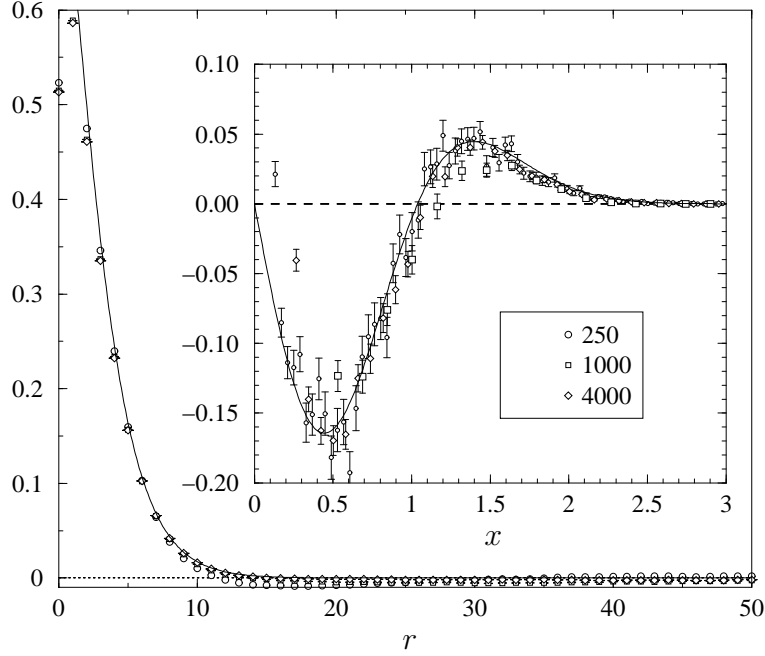


Figure 4: Spin-spin correlation function. In the inlay the rescaled functions $\sqrt{n}G_n^{\text{ss}}(r)$ are plotted as the function of the scaling variable $x = \frac{r}{2\sqrt{n}}$. The lines denote the theoretical predictions respectively for non-scaling and scaling parts.

non-scaling part of the expression (38) in the $n \rightarrow \infty$ limit. As we increase the size we clearly see the emerging signal of the short range correlations which approaches the theoretical curve.

Matter-geometry interactions

At the first sight the appearance of the long-range component in correlation functions of non-critical fields is surprising. It is however clear that it is mediated through the geometry. To see this more explicitly we consider an example of fields non-interacting with themselves : $g_{q,x;q',x'} = 1$ but interacting with the geometry through the term $p_{q,x}$. This example can be easily solved by introducing new weights and operators :

$$\tilde{p}_q = \sum_x p_{x,q} \quad \text{and} \quad \tilde{A}_q = \frac{\sum_x A_x p_{q,x}}{\sum_x p_{q,x}}. \quad (48)$$

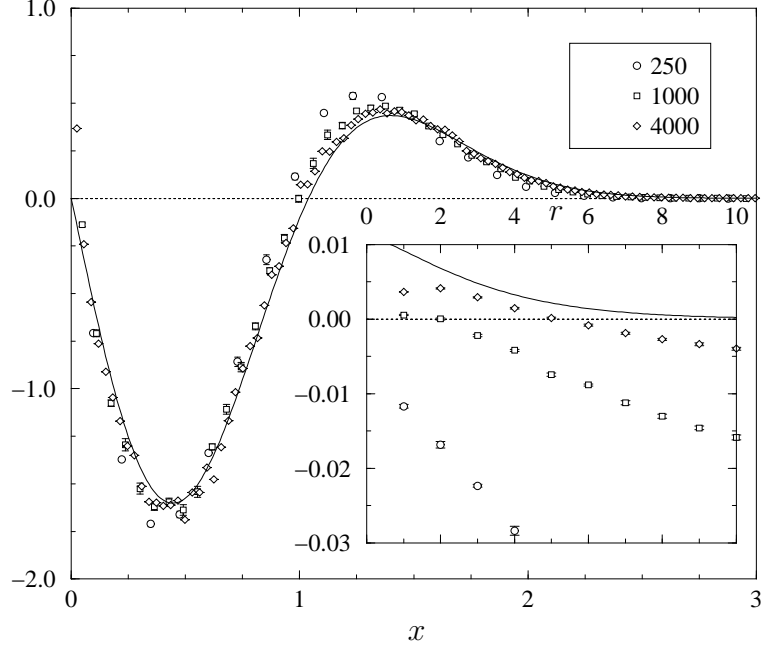


Figure 5: “Curvature-curvature” correlation function. In the main part rescaled functions $\sqrt{n} G_n^{\mathbf{q}\mathbf{q}}(r)$ are plotted as the function of the scaling variable $x = \frac{r}{2\sqrt{n}}$. The inlay shows unmodified functions. The lines denote the theoretical predictions respectively for scaling and non-scaling parts

The correlation functions of the operators \tilde{A}_q with new weight \tilde{p} are identical to the correlation functions of the operator A_x with old weights p . We see that in this case matter operators behave as geometric operators and although matter is non-interacting they pick up long range correlations through coupling to geometry.

To see if the effects described in previous section can be explained by the above mechanism we assume that the spins are independent and interact only with the geometry. The effective coupling term $p_{q,s}$ can be read from (16) :

$$p_{q,s} = \frac{e^{sh} Z_{q,s}^{q-1}(\mu_0) \Delta Z_s(\mu_0)}{\sum_s e^{sh} F(Z_s(\mu_0)) \Delta Z_s(\mu_0)}. \quad (49)$$

The effective shift is then given by :

$$\tilde{\delta}_s^{eff} = \frac{1}{2} \frac{\langle \mathbf{q} \rangle}{\langle \mathbf{q}^2 \rangle - \langle \mathbf{q} \rangle^2} \frac{\langle \mathbf{s}\mathbf{q} \rangle - \langle \mathbf{q} \rangle \langle \mathbf{s} \rangle}{\langle \mathbf{s} \rangle}. \quad (50)$$

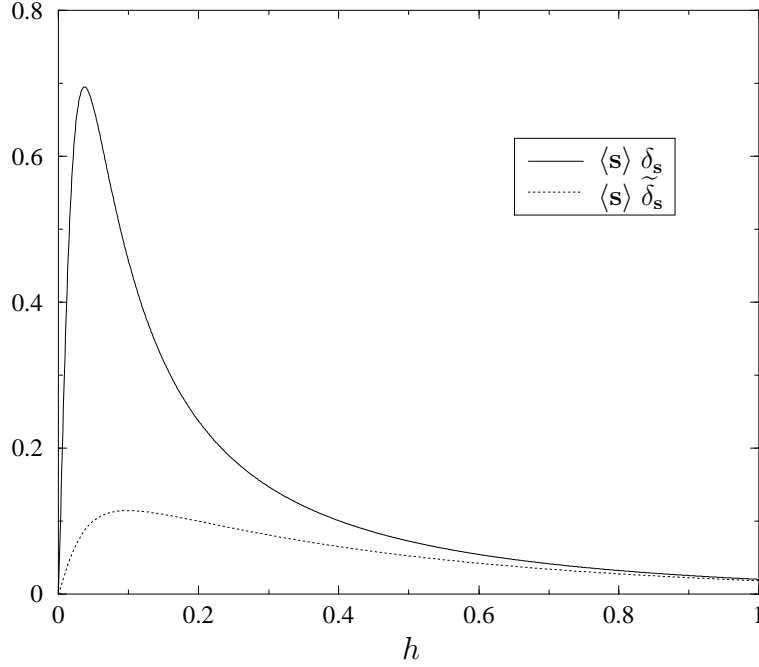


Figure 6: Comparison of interacting (continuous line) and non-interacting (dotted line) spins.

On the figure 6 we have plotted for comparison⁵ $\langle \mathbf{s} \rangle \cdot \delta_{\mathbf{s}}$ and $\langle \mathbf{s} \rangle \cdot \tilde{\delta}_{\mathbf{s}}$. While we see a clear correlation, there is no quantitative agreement. The real effect is much more pronounced. This is due to the non-zero spin-spin correlation length. As a results we have an area-area like interactions rather than point-point like as we assumed. When h is increased the correlation length decreases and the two values begin to agree as expected (see figure 6). Probably the proper way to proceed in this case is to use renormalization group analysis. Blocking the spins would eventually leave us with a non-interacting model and one could expect that the shift would be then given by the formula (50). It is so far unclear how to perform this blocking on the BP geometry. However a prescription that works for Ising model on 2D random surfaces could provide a testing ground for this hypothesis[16].

⁵ $\langle \mathbf{s} \rangle \cdot \delta_{\mathbf{s}} \approx \delta_{\phi}^{eff}$. where $\phi = (s+1)/2$. This corresponds to the definition used in [3].

Discussion

We have analyzed in detail a simple model of random geometry and derived the general structure of the correlation functions. We observe that in general every connected correlator contains a weak-long-range scaling term. This term is present even in the correlators of non-critical matter fields. The appearance of this term is a direct consequence of the asymptotic behavior (24) of the disconnected correlation functions. This behavior has a geometric origin and is picked up by the matter fields that effectively couple locally to the geometry. As we have shown in the appendix C this coupling is suppressed in the presence of the symmetry. This is intuitively clear : When $h = 0$ the probability of the spin being up or down must be equal independently of the surrounding geometry. When $h \neq 0$ probability of spin being up is dependent on the vertex number q . This can be seen for example from the fact that in this case $\langle \mathbf{s} \mathbf{q} \rangle \neq \langle \mathbf{s} \rangle \langle \mathbf{q} \rangle$.

The long-range scaling term can be strongly suppressed by the use of a slightly modified definition of connected correlator (32). This form of improved correlator postulated in [3] is here supported by analytical calculations.

Conversely also purely geometric correlators pick up the non-scaling behavior coming from the matter-matter interactions. In case of the model studied here this effect is very small and for non-critical matter it is irrelevant in scaling limit. Nevertheless we consider this as an interesting mechanism that could play a role in some more complex models which would allow for the existence of the critical matter fields.

While BP provide only a toy model of random geometry, numerical simulations indicate that the asymptotic behavior (24) is valid in almost every kind of random geometry ensembles. In particular the behavior of the Ising model in a magnetic field coupled to the 2D simplicial quantum gravity exhibits qualitatively the same features and the improved correlator can be successfully used [3]. Similar structure is also observed for the correlators of the local action density of the Abelian gauge fields in the 4D simplicial quantum gravity [17].

It thus seems that there exists a high degree of “universality” in the structure of correlators in systems with a random geometry. Simple models as the one described above give us so far unique opportunity to study this structure analytically.

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A Critical point

Assuming the expansion :

$$Z_{q,x}(\mu_0 + \Delta\mu) \approx Z_{q,x}(\mu_0) - \Delta Z_{q,x}^{(1)}(\mu_0) \sqrt{\Delta\mu} + \Delta Z_{q,x}^{(2)}(\mu_0) \Delta\mu \quad (51)$$

and inserting it into (5) we obtain following equations :

$$\sum_{q',x'} (\delta_{q,x;q',x'} - M_{q,x;q',x'}) \Delta Z_{q',x'}^{(1)} = 0, \quad (52)$$

$$\begin{aligned} Z_{q,x} = & - \sum_{q',x'} (\delta_{q,x;q',x'} - M_{q,x;q',x'}) \Delta Z_{q',x'}^{(2)} \\ & + \frac{1}{2} \sum_{q',x'} g_{q,x;q',x'} p_{q',x'} (q' - 1)(q' - 2) Z_{q',x'}^{q'-3} (\Delta Z_{q',x'}^{(1)})^2. \end{aligned} \quad (53)$$

Equation (52) states that $\Delta Z_{q,x}^{(1)}$ is an eigenvector of the matrix $\mathbf{M}(\mu_0)$ associated with the eigenvalue $\lambda^{(0)} = 1$. We thus have $\Delta Z_{q,x}^{(1)} = \mathcal{C} v_{q,x}^{(0)}$. Denoting by $V_{q,x}^{(0)}$ the left eigenvector of matrix $\mathbf{M}(\mu)$ associated with the eigenvalue $\lambda^{(0)} = 1$ we obtain from (53) :

$$\begin{aligned} \sum_{q,x} V_{q,x}^{(0)} Z_{q,x} = & \frac{1}{2} \mathcal{C}^2 e^{-\mu_0} \sum_{q,x} \sum_{q',x'} V_{q,x}^{(0)} \\ & g_{q,x;q',x'} p_{q',x'} (q' - 1)(q' - 2) Z_{q',x'}^{q'-3} (v_{q',x'}^{(0)})^2. \end{aligned} \quad (54)$$

From this we can easily calculate constant \mathcal{C} .

B Transfer matrix

Matrix $\widetilde{\mathbf{M}}$ is not symmetric and it's eigenvectors are not orthogonal. It is more convenient to work with a symmetric matrix. To this end we define the transfer matrix $\widetilde{\mathbf{T}}$ by :

$$\widetilde{T}_{q,x;q',x'}(\mu) = \frac{f_{q,x}}{f_{q',x'}} M_{q,x;q',x'}(\mu) \quad q, q' > 1 \quad (55)$$

and zero otherwise, where

$$f_{q,x} = \sqrt{(q-1)p_{q,x}Z_{q,x}^{q-2}(\mu)}. \quad (56)$$

The matrix $\mathbf{T}(\mu)$ is symmetric, has the same set of eigenvalues $\lambda^{(k)}$ as matrix $\widetilde{\mathbf{M}}(\mu)$ and its eigenvectors $\mathbf{u}^{(k)}$ are related to the eigenvectors $\widetilde{\mathbf{v}}^{(k)}$ of the matrix $\widetilde{\mathbf{M}}(\mu)$ by :

$$\widetilde{v}_{q,x}^{(k)} = \begin{cases} \frac{1}{f_{q,x}} u_{q,x}^{(k)}, & q > 1. \end{cases} \quad (57)$$

In terms of matrix $\widetilde{\mathbf{T}}$ correlation functions can be rewritten as :

$$G_{\mu}^{\mathbf{AB}}(r) = \sum_{q,x} \sum_{r>1,y} \sum_{q',x'} \sum_{r'>1,y'} \frac{Z_{q,x}(A)}{f_{r,y}} M_{q,x;r,y} (\widetilde{T}^{r-2})_{r,y;r',y'} (M^T)_{r',y';q',x'} \frac{Z_{q',x'}(B)}{f_{r',y'}}. \quad (58)$$

This can be further written in terms of an orthonormal basis of eigenvectors $\mathbf{u}_{q,x}^{(k)}$:

$$G_{\mu}^{\mathbf{AB}}(r) = \sum_k \lambda_k^{r-2} \sum_{q,x} \sum_{r>1,y} Z_{q,x}(\mathbf{A}) M_{q,x;r,y} \frac{u_{r,y}^{(k)}}{f_{r,y}} \sum_{q',x'} \sum_{r'>1,y'} Z_{q',x'}(\mathbf{B}) M_{q',x';r',y'} \frac{u_{r',y'}^{(k)}}{f_{r',y'}}. \quad (59)$$

Using (57) and the relation between eigenvectors v of matrix \mathbf{M} and $\widetilde{\mathbf{M}}$:

$$v_{q,x}^{(k)} = \begin{cases} \widetilde{v}_{q,x}^{(k)} & q > 1 \\ \frac{1}{\lambda_k} \sum_{q'>1,x'} M_{1,x;q',x'} \widetilde{v}_{q',x'}^{(k)} & q = 1 \end{cases} \quad (60)$$

we finally obtain the expression (20).

C Symmetry

Let us assume that matter field takes values in a group and that the weights are invariant under the action of this group :

$$g_{q,y;x;q',y,x'} = g_{q,x;q',x'} \quad \text{and} \quad p_{q,y;x} = p_{q,x}. \quad (61)$$

From the above and the definition (1) we immediately obtain that $Z_{q,x}(\mu) = Z_{q,1}(\mu)$ and so also $\Delta Z_{q,x}^{(1)}(\mu_0) = \Delta Z_{q,1}^{(1)}(\mu_0)$. As we have shown in appendix A this implies $v_{q,y}^{(0)} = v_{q,1}^{(0)}$.

If an operator A_x depends only on the values of the matter fields then :

$$\sum_{q,x} Z_{q,x}(A_x - \langle A \rangle) v_{q,x}^{(0)} = \sum_q Z_{q,1}(\mu) p_q v_{q,1}^{(0)} \sum_x (A_x - \langle \mathbf{A} \rangle) \quad (62)$$

But from (16) we have :

$$\langle \mathbf{A} \rangle = \frac{\sum_q p_{q,1} Z_{q,1}^{q-1}(\mu_0) \Delta Z_{q,1}(\mu_0) \sum_x A_x}{\sum_q p_{q,1} Z_{q,1}^{q-1}(\mu_0) \Delta Z_{q,1}(\mu_0) \sum_x 1} = \frac{\sum_x A_x}{\sum_x 1} \quad (63)$$

which inserted into (62) immediately leads to :

$$\sum_{q,x} Z_{q,x}(\mathbf{A} - \langle \mathbf{A} \rangle) v_{q,x}^{(0)} \equiv 0. \quad (64)$$

This ensures that the long-range term in (20) corresponding to the eigenvalue $\lambda^{(0)} = 1$ will be absent for any correlation function of the type $G^{(\mathbf{A} - \langle \mathbf{A} \rangle) \mathbf{B}}(r)$.

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